# Serbian Mathematical Olympiad 2016

**for high school students**

Belgrade, April 1–2

**Problems and Solutions**

*Edited by Dušan Djukić*

*Cover photo: Golubac fortress*

# SHORT HISTORY AND SYSTEM

Mathematical competitions in Serbia have been held since 1958. In the first years only republic competitions within the former Yugoslavia, which Serbia was a part of, were held. The first Federal Mathematical Competition in Yugoslavia was held in Belgrade in 1960, and since then it was held regularly every year, skipping only 1999 for a non-mathematical reason. The system has undergone relatively few changes. The earliest Federal Competitions were organized for 3rd and 4th grades of high school only; 2nd grade was added in 1970, and 1st grade in 1974. Since 1982, 3rd and 4th grades compose a single category. After the breakdown of the old Yugoslavia in 1991, the entire system was continued in the newly formed FR Yugoslavia, later renamed Serbia and Montenegro. The separation of Montenegro finally made the federal competition senseless as such. Thus, starting with 2007, the federal competition and team selection exam are replaced by a two-day Serbian Mathematical Olympiad.

Today a mathematical competition season in Serbia consists of four rounds of increasing difficulty:

- Municipal round, held in December. The contest consists of 5 problems, each 20 points worth, to be solved in 3 hours. Students who perform well qualify for the next round (50-60 points are usually enough).

- Regional round, held in late January in the same format as the municipal round. Each student's score is added to that from the municipal round. Although the number of students qualified for the state round is bounded by regional quotas, a total score of 110-120 should suffice.

- State (republic) round, held in March in a selected town in the country. There are roughly 200 to 300 participants. The contest consists of 5 problems in 4 hours.

- Serbian Mathematical Olympiad (SMO), held in early April in a selected place in the country. The participants are selected through the state round: 28 from A category (distribution among grades: 4+8+8+8), 3 from B category (0+0+1+2), plus those members of the last year's olympic team who did not manage to qualify otherwise. Six most successful contestants are invited to the olympic team.

Since 1998, contests for each grade on the three preliminary rounds are divided into categories A (specialized schools and classes) and B (others). A student from category B is normally allowed to work the problems for category A instead. On the SMO, all participants work on the same problems.

The 10-th Serbian Mathematical Olympiad (SMO) for high school students took place in Belgrade on April 1–2, 2016. There were 31 students from Serbia and 6 guest students - 4 from Russia and 2 from Republika Srpska (Bosnia and Herzegovina). The average score on the contest was 9.07 points. Problems 1 and 4 were relatively easy, but few students solved problems 2, 3 and 5, and nobody solved problem 6.

The team for the 33-rd Balkan MO and 57-th IMO was selected based on the SMO and an additional team selection test:

| | | |
|---|---|---|
| Igor Medvedev | Math High School, Belgrade | 20 points |
| Nikola Pavlović | HS "Jovan Jovanović Zmaj", Novi Sad | 19 points |
| Aleksa Milojević | Math High School, Belgrade | 16 points |
| Ognjen Tošić | Math High School, Belgrade | 14 points |
| Aleksa Konstantinov | Math High School, Belgrade | 14 points |
| Nikola Sadovek | Math High School, Belgrade | 14 points |

In this booklet we present the problems and full solutions of the Serbian Mathematical Olympiad, team selection test and the Balkan Mathematical Olympiad.

**Serbian MO 2016 – Problem Selection Committee**

- Vladimir Baltić
- Bojan Bašić *(chairman)*
- Dušan Djukić
- Miljan Knežević
- Nikola Petrović
- Marko Radovanović

# SERBIAN MATHEMATICAL OLYMPIAD

**for high school students**

**Belgrade , 01.04.2016.**

## First Day

**1.** Let $n$ be a positive integer greater than 1. Prove that there exists a positive integer $m$ greater than $n^n$ such that

$$\frac{n^m - m^n}{n+m}$$

is a positive integer. *(Nikola Petrović)*

**2.** Given a positive integer $n$, define $f(0,j) = f(i,0) = 0$, $f(1,1) = n$ and

$$f(i,j) = \left\lfloor \frac{f(i-1,j)}{2} \right\rfloor + \left\lfloor \frac{f(i,j-1)}{2} \right\rfloor$$

for all positive integers $i$ i $j$, $(i,j) \neq (1,1)$. How many ordered pairs of positive integers $(i,j)$ are there for which $f(i,j)$ is an odd number? *(Dušan Djukić)*

**3.** Let $O$ be the circumcenter of triangle $ABC$. A tangent $t$ to the circumcircle of triangle $BOC$ meets the sides $AB$ and $AC$ at points $D$ and $E$, respectively ($D, E \not\equiv A$). Point $A'$ is the reflection of $A$ in line $t$. Prove that the circumcircles of triangles $A'DE$ and $ABC$ are tangent to each other. *(Dušan Djukić)*

Time allowed: 270 minutes.
Each problem is worth 7 points.

# SERBIAN MATHEMATICAL OLYMPIAD

**for high school students**

**Belgrade, 02.04.2016.**

## Second Day

**4.** In a triangle $ABC$ $(AB \neq AC)$, the incircle with center $I$ is tangent to side $BC$ at point $D$. Let $M$ be the midpoint of side $BC$. Prove that the perpendiculars from points $M$ and $D$ to lines $AI$ and $MI$ respectively meet on the altitude in $\triangle ABC$ from $A$ or its extension. *(Dušan Djukić)*

**5.** Given any $2n-1$ two-element subsets of set $\{1, 2, \ldots, n\}$, prove that one can always choose $n$ of these subsets such that their union contains at most $\frac{2}{3}n + 1$ elements. *(Dušan Djukić)*

**6.** Suppose $a_1, a_2, \ldots, a_{2^{2016}}$ are positive integers such that, for all $n$ with $1 \leqslant n \leqslant 2^{2016}$,

$$a_n \leqslant 2016 \quad \text{and} \quad a_1 a_2 \cdots a_n + 1 \text{ is a perfect square.}$$

Prove that at least one of the numbers $a_1, a_2, \ldots, a_{2^{2016}}$ must be equal to 1. *(Dušan Djukić)*

Time allowed: 270 minutes.
Each problem is worth 7 points.

## SOLUTIONS

**1.** We start by noting that if $m > n \geqslant 3$, then $n^m > m^n$ and hence $\frac{n^m - m^n}{m+n} > 0$. Indeed, the function $f(x) = \frac{\ln x}{x}$ is decreasing for $x > e$ because $f'(x) = \frac{1-\ln x}{x^2} < 0$, so $\frac{\ln n}{n} > \frac{\ln m}{m}$, i.e. $m \ln n > n \ln m$, which yields $n^m = e^{m \ln n} > e^{n \ln m} = m^n$.

For $n = 2$ we can choose $m = 10$. Suppose that $n > 2$. We have

$$n^m - m^n \equiv n^m - (-n)^n = n^n(n^{m-n} - (-1)^n) \pmod{m+n}.$$

We shall look for $m$ in the form $m = kn^n - n$ ($k \in \mathbb{N}$). Then $m + n = kn^n \mid n^m - m^n$ if and only if $k \mid n^{m-n} - (-1)^n$.

(1°) If $n$ is odd, then $n^{m-n} - (-1)^n$ is even, so we can take $k = 2$ and $m = 2n^n - n$.

(2°) If $n$ is even, then $n^{m-n} - (-1)^n = n^{m-n} - 1$ is divisible by $n - 1$, so we can take $k - n - 1$ and $m = (n-1)n^n - n$.

*Remark.* The inequality $n^m > m^n$ for $m > n \geqslant 3$ can also be easily proved by induction.

There are many possible choices for $m$. For instance, for $n > 2$ one can take $m = pn^n - n$, where $p$ is any prime divisor of $n^{n^n - 2n} - (-1)^n$.

**2.** The answer is $n$.

Denote $s_m = \sum_{i+j=m} f(i,j)$ for $m \geqslant 2$. Since $f(i,j) - 2[\frac{f(i,j)}{2}]$ equals 1 if $f(i,j)$ is odd and 0 otherwise, the number of odd terms among $f(i,j)$ for $i, j \geqslant 0$ with $i + j = m$ is

$$\begin{aligned}\textstyle\sum_{i+j=m} \left(f(i,j) - 2\left[\frac{f(i,j)}{2}\right]\right) &= s_m - \textstyle\sum_{i+j=m} \left(\left[\frac{f(i-1,j+1)}{2}\right] + \left[\frac{f(i,j)}{2}\right]\right) \\ &= s_m - s_{m+1}.\end{aligned}$$

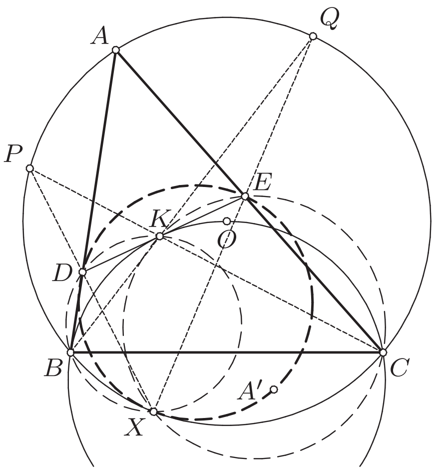
It follows that the number of pairs $(i,j)$ for which $f(i,j)$ is odd and $i + j < m$ equals $s_2 - s_m = n - s_m$.

All that remains is to show that $s_m = 0$ if $m$ is big enough. Clearly, the sequence $s_m$ is nonnegative and non-increasing, so there exist $N$ and $k$ such that $s_m = k$ for all $m \geqslant N$. This means that $f(i,j)$ is even whenever $i + j \geqslant m$. Suppose that $k > 0$ and consider the smallest $i$ for which $f(i, m-i) > 0$. A simple induction yields $f(i, m + r - i) = [\frac{f(i,m-i)}{2^r}]$ for $r \geqslant 1$. However, if $r$ is taken so that $2^r \leqslant f(i, m-i) < 2^{r+1}$, this would imply $f(i, m + r - i) = 1$, contrary to the assumption.

*Second solution (U. Dinić).* Consider the following game. Initially, there are $n$ markers at point $(1,1)$ in the coordinate plane. A step consists of simultaneously performing the following moves for all $i, j$: If there are $m$ markers at point $(i, j)$, move exactly $[m/2]$ of these markers to each of the points $(i+1, j)$ and $(i, j+1)$, thus leaving one marker at $(i, j)$ if $m$ is odd, and no markers if $m$ is even. Observe that a row or column that is nonempty at some moment can never be emptied. Thus no marker can leave the square $[1, n] \times [1, n]$, which means that the game will end in a finite time.

It is easy to see that after $i + j - 2$ steps the number of markers at point $(i, j)$ equals $f(i, j)$, and that in the final position there is a marker at $(i, j)$ if and only if $f(i, j)$ is odd.

**3.** Denote by $K$ the tangency point of the line $t$ and circle $BOC$. Let the circumcircles of triangles $BDK$ and $CEK$ meet at point $X \neq K$. Since $\sphericalangle BXC = \sphericalangle BXK + \sphericalangle KXC = \sphericalangle ADK + \sphericalangle KEA = 180^\circ - \sphericalangle CAB$, point $X$ lies on the circumcircle $k$ of triangle $ABC$. Moreover, $\sphericalangle DXE = \sphericalangle DXK + \sphericalangle KXE = \sphericalangle DBK + \sphericalangle KCE = \sphericalangle CKB - \sphericalangle CAB = \sphericalangle CAB = \sphericalangle DA'E$, so $X$ also lies on the circumcircle $k_1$ of triangle $A'DE$. We shall prove that the circles $k$ and $k_1$ touch at point $X$.

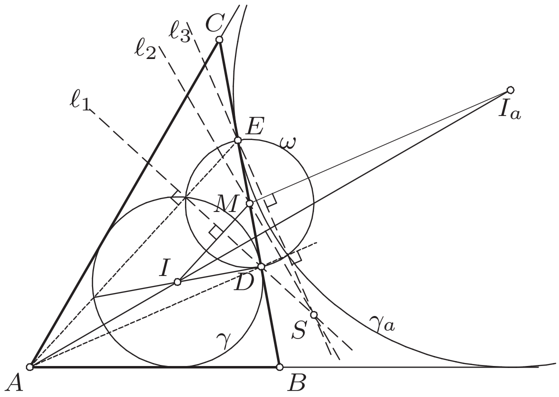



If $P$ is the intersection point of the lines $CK$ and $XD$, then $\sphericalangle XPC = \sphericalangle XDE - \sphericalangle CKE = \sphericalangle XBK - \sphericalangle CBK = \sphericalangle XBC$, which means that $P$ lies on circle $k$. Analogously, the lines $BK$ and $XE$ intersect at point $Q$ on circle $k$. Finally, $PQ \parallel DE$ because $\sphericalangle XPQ = \sphericalangle XBQ = \sphericalangle XDK$. Therefore the triangles $XDE$ and $XPQ$ are homothetic with the center of homothety $X$, so their circumcircles touch at $X$.

<u>*Second solution.*</u> Let the lines $BK$ and $CK$ meet the circumcircle of $\triangle ABC$ again at points $Q$ and $P$, respectively. Since $\sphericalangle CPQ = \sphericalangle CBQ = \sphericalangle CKE$, we have $PQ \parallel DE$. Let the lines $DP$ and $EQ$ meet at point $X$. Since the points $D = PX \cap AB$, $K = PC \cap QB$ and $E = AC \cap QX$ collinear, the converse Pascal theorem implies that $X$ lies on the same conic (circle) with points $A, B, C, P, Q$. Hence the triangles $XDE$ and $XPQ$ are homothetic, so their circumcircles are tangent to each other at the center of homothety $X$. Finally, point $A'$ lies on the circumcircle of $\triangle DEX$ because $\sphericalangle DXE = \sphericalangle PXQ = \sphericalangle PCA + \sphericalangle ABQ = \sphericalangle BKC - \sphericalangle BAC = \sphericalangle BAC = \sphericalangle DA'E$.

<u>*Remark.*</u> More generally: If $K$ is an arbitrary point inside $\triangle ABC$, the tangent at $K$ to the circle $BKC$ meets $AB$ at $D$ and $AC$ at $E$, and the circles $BDK$ and $CEK$ meet again at $X$, then the circles $DEX$ and $ABC$ are tangent at $X$.

**4.** Denote by $\gamma$ the inircle of $\triangle ABC$, and by $\gamma_a$ and $I_a$ the excenter across $A$ and its center. Circle $\gamma_a$ touches side $BC$ at point $E$ symmetric to $D$ about $M$. The perpendicular $\ell_1$ from $D$ to $MI$ is the radical axis of $\gamma$ and circle $\omega$ with diameter $DE$, while the perpendicular $\ell_2$ from $M$ to $AI$ is the radical axis of circles $\gamma$ and $\gamma_a$ (as $MD = ME$). Moreover, the radical axis of $\gamma_a$ and $\omega$ is the perpendicular $\ell_3$ from $E$ to $MI_a$. The lines $\ell_1, \ell_2$ and $\ell_3$ meet at the radical center $S$ of the circles $\gamma, \gamma_a, \omega$. On the other hand, it is well known that $MI \parallel AE$ and $MI_a \parallel AD$, so the lines $\ell_1$ and $\ell_3$ contain the altitudes from $D$ and $E$ in triangle $ADE$. Therefore $S$ is the orthocenter of $\triangle ADE$, which clearly lies on the altitude from $A$.




<u>Second solution.</u> Let $S$ be the intersection of the perpendiculars from $M$ and $D$ to $AI$ and $MI$ respectively, and let the line $MI$ intersect the altitude from $A$ in $\triangle ABC$ at point $J$. It suffices to show that $AJ = ID$. Indeed, this would imply that $AJDI$ is parallelogram, so $MS \perp DJ$ and hence $D$ is the orthocenter of $\triangle MSJ$, which in turn implies $JS \perp MD$, i.e. $AS \perp BC$.

This can be done by a straightforward computation. Denoting by $H$ and $F$ respectively the feet of the altitude and angle bisector from $A$ and $a = BC$, $b = CA$, $c = AB$, we have $BF = \frac{ac}{b+c}$, $BD = \frac{a-b+c}{2}$ i $BH = \frac{a^2-b^2+c^2}{2a}$, so $FH = BF - BH = \frac{|b-c|((b+c)^2-a^2)}{2a(b+c)}$, $FD = BF - BD = \frac{|b-c|(b+c-a)}{2(b+c)}$ and therefore $\frac{AJ}{AH} = \frac{FD}{FH} = \frac{a}{a+b+c} = \frac{ID}{AH}$.

**5.** We shall prove by induction on $k$ ($k \leqslant \frac{2n-1}{3}$) that one can always remove $3k$ subsets such that the cardinality of the union of the remaining subsets does not exceed $n-k$.

The case $k = 0$ is trivial. Assume that $k \geqslant 1$ and that we have removed $3(k-1)$ subsets so that the union of the remaining ones has at most $n-k+1$ elements. Since $2n-1-3(k-1) < 2(n-k+1)$, there is an element $x_k$ from the union that is contained in at most three of the remaining subsets. Then we can remove three subsets so that the union of the remaining $2n-1-3k$ subsets does not contain $x_k$, which finishes the induction.

The problem statement follows for $k = [\frac{n-1}{3}]$ as $n - [\frac{n-1}{3}] \leqslant n - \frac{n-3}{3} = \frac{2}{3}n + 1$.

<u>Remark.</u> As tempting as it might be, the probabilistic method does not seem to easily yield the desired bound.

**6.** The key observation is that, if $a+1 = u^2$ and $b = v^2$ are perfect squares and $a > b$, then $ab+1$ is not a perfect square. Indeed, $(uv-1)^2 < ab+1 = u^2v^2-v^2+1 < (uv)^2$.

Let $p_1, p_2, \ldots, p_m$ be all prime numbers less than 2016. For $1 \leqslant n \leqslant 2^{2016}$ consider the binary sequence $\mathrm{c}_n = (r_1, r_2, \ldots, r_m)$, where $r_i = 0$ if the exponent at $p_i$ in the product $P_n = a_1 a_2 \cdots a_n$ is even, and $r_i = 1$ otherwise. Note that there are only $2^m$ possibilities for the sequence $\mathrm{c}_n$. Thus for every $k \leqslant 2^{2016} - 2^m$ there exist indices $s$ and $t$ with $k \leqslant s < t \leqslant k + 2^m$ such that $\mathrm{c}_s = \mathrm{c}_t$, and then $P_t/P_s$ is a perfect square not exceeding $2016^{2^m}$.

Suppose that no term of the sequence $(a_n)$ equals 1. Take $k = 11 \cdot 2^m$; then clearly $k+2^m < 2^{2016}$. As we have seen, for some $s, t$ with $k \leqslant s < t \leqslant k+2^m$ the quotient $b = P_t/P_s$ is a perfect square, but $a = P_s \geqslant P_k \geqslant 2^k = 2048^{2^m} > 2016^{2^m} \geqslant b$, so by the above observation, $a + 1 = P_s + 1$ and $ab + 1 = P_t + 1$ cannot both be perfect squares, a contradiction.

# Additional Team Selection Test

**Belgrade, 05.04.2016.**

**1.** A sequence of polynomials $P_n(x)$ is given by

$$P_0(x) = x^3 - 4x \quad \text{and} \quad P_{n+1}(x) = P_n(1+x)P_n(1-x) - 1.$$

Prove that the polynomial $P_{2016}(x)$ is divisible by $x^{2016}$. *(Dušan Djukić)*

**2.** Let $ABCD$ be a square of side 4. Determine the largest positive integer $k$ with the following property: For an arbitrary arrangement of $k$ points strictly inside square $ABCD$, one can always find a square of side 1, entirely contained in square $ABCD$ (with sides not necessarily parallel to the sides of square $ABCD$), whose strict interior contains none of the $k$ given points. *(Bojan Bašić)*

**3.** Denote by $w(x)$ the largest odd divisor of a positive integer $x$. Suppose that $a$ and $b$ are coprime positive integers such that $a + w(b+1)$ and $b + w(a+1)$ are powers of two. Prove that $a+1$ and $b+1$ are powers of two. *(Dušan Djukić)*

Time allowed: 270 minutes.
Each problem is worth 7 points.

## SOLUTIONS

**1.** For $n \geqslant 1$, the recurrence relation implies that $P_n$ is an even function and

$$\begin{aligned} P_{n+2}(x) &= \left(P_n(2+x)P_n(-x)-1\right)\left(P_n(2-x)P_n(x)-1\right)-1 \\ &= P_n(2+x)P_n(2-x)P_n^2(x)-\left(P_n(2+x)+P_n(2-x)\right)P_n(x), \end{aligned}$$

hence $P_n \mid P_{n+2}$. We also observe that polynomial $P_n(2+x)+P_n(2-x)$ is divisible by $x$ (and in turn by $x^2$, being even) if $x-2 \mid P_n(x)$. Therefore, if $x^k(x-2) \mid P_n(x)$ for some $k \geqslant 2$, then $x^{k+2}(x-2) \mid P_{n+2}(x)$.

Note that $P_0$ is an odd function and $P_2(x) = P_0(x+2)P_0(x-2)P_0(x)^2+(P_0(x+2)+P_0(x-2))P_0(x)$, implying that $x^2(x-2) \mid P_2(x)$. Now a simply induction yields $x^n(x-2) \mid P_n(x)$ whenever $2 \mid n$ ($n \in \mathbb{N}$).

**2.** The answer is $k = 15$.

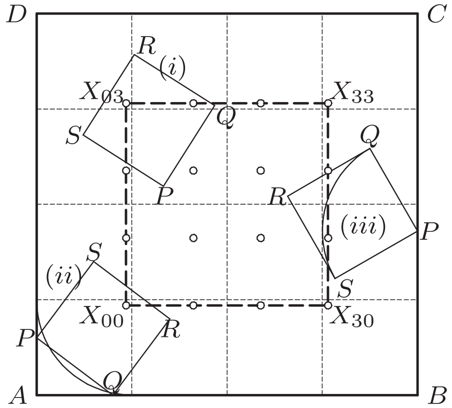
It is clear that $k = 15$ fulfils the requirements. Indeed, divide square $ABCD$ into 16 unit squares; given any 15 points inside square $ABCD$, at least one of these unit square contains no given points.

We now show that $k < 16$. In the square with the vertices $A(-2,-2)$, $B(2,-2)$, $C(2,2)$, $D(-2,2)$, mark the 16 points $X_{ij}(-a+i\cdot\frac{2a}{3}, -a+j\cdot\frac{2a}{3})$ for $i,j \in \{0,1,2,3\}$, where $1 < a < \frac{3}{2\sqrt{2}}$. In order to prove that every unit square $PQRS$ contains at least one marked point, we shall use the following simple statement.

*Lemma.* If $E$ and $F$ respectively are points on the sides $BC$ and $CD$ of a unit square $ABCD$ such that $d(A,EF) = 1$, then $\sqrt{8}-2 \leqslant EF \leqslant 1$.

*Proof.* Circle $(A, AB)$ is tangent to the segment $EF$ at some point $G$; we have $EF = GE + GF = BE + DF$ and hence $CE + CF + EF = 2$. Since $EF \leqslant CE + CF \leqslant \sqrt{2}EF$, the statement follows. $\square$

It suffices to consider the following three cases.




(i) The center $O$ of square $PQRS$ is inside the square $X_{00}X_{03}X_{33}X_{30}$. Then at least one of the points $X_{ij}$ is on a distance from $O$ not exceeding $\frac{a\sqrt{2}}{2} < \frac{1}{2}$; this point is inside square $PQRS$.

(ii) All four vertices $P, Q, R, S$ are outside the square $X_{00}X_{03}X_{33}X_{30}$. Suppose without loss of generality that $P \in AD$ and $Q \in AB$, and consider point $W(-1,-1)$. The lemma implies that

$d(X_{00}, PQ) < d(W, PQ) \leqslant 1$, so $X_{00}$ is within square $PQRS$.

(iii) exactly one of the vertices, say $R$, lies inside $X_{00}X_{03}X_{33}X_{30}$. Suppose without loss of generality that $P$ is on side $BC$. Since $d(P, X_{30}X_{33}) < 1$, the lemma implies that the portion of line $X_{30}X_{33}$ inside square $PQRS$ has a length greater than $\sqrt{8} - 2 > a$, so it contains at least one marked point.

**3.** As usual, we write $2^r \parallel x$ when $2^r \mid x$ and $2^{r+1} \nmid x$. We call a pair $(a, b)$ a $(k, l)$-*pair* if $a + w(b+1)$ and $b + w(a+1)$ are powers of two and $2^k \parallel a+1$ and $2^l \parallel b+1$.

Consider a $(k, l)$-pair $(a, b)$; let $a = 2^k c - 1$, $b = 2^l d - 1$ and

$$a + w(b+1) = 2^k c + d - 1 = 2^m \quad \text{and} \quad b + w(a+1) = 2^l d + c - 1 = 2^n. \qquad (*)$$

If $c = 1$, then also $d = 1$ (and vice-versa), so $(a, b) = (2^k - 1, 2^l - 1)$.

Suppose that $c, d > 1$. It follows from $(*)$ that $2^k \parallel d-1 = 2^k b'$ and $2^l \parallel c-1 = 2^l a'$ for some odd $a', b'$; when substituted in $(*)$, these yield $2^l a' + b' + 1 = 2^{m-k}$ and $2^k b' + a' + 1 = 2^{n-l}$. From this we get $2^k \parallel a'+1$ and $2^l \parallel b'+1$, so the above relations become

$$a' + w(b'+1) = a' + \frac{b'+1}{2^l} = 2^{m-k-l} \quad \text{and analogno} \quad b' + w(a'+1) = 2^{n-k-l}.$$

Thus $(a', b') = \left(\frac{a+1-2^k}{2^{k+l}}, \frac{b+1-2^l}{2^{k+l}}\right)$ is also a $(k, l)$-pair, but $a' < a$ and $b' < b$.

Define $a_1 = a$, $b_1 = b$ and $a_{n+1} = \frac{a_n+1-2^k}{2^{k+l}}$, $b_{n+1} = \frac{b_n+1-2^l}{2^{k+l}}$. By the above consideration, each pair $(a_i, b_i)$ is a $(k, l)$-pair, and $a_n = 2^k - 1$ and $b_n = 2^l - 1$ for some $n$. Going backwards, we easily find that

$$a = \frac{2^{n(k+l)} - 1}{2^{k+l} - 1}(2^k - 1) \quad \text{and} \quad b = \frac{2^{n(k+l)} - 1}{2^{k+l} - 1}(2^l - 1).$$

Since $a$ and $b$ are coprime, we finally infer that $n = 1$, so $a = 2^k - 1$ and $b = 2^l - 1$.

The 33-rd Balkan Mathematical Olympiad was held from May 5 to May 10 in Tirana in Albania. The results of the Serbian contestants are shown below:

| | 1 | 2 | 3 | 4 | Total | |
|---|---|---|---|---|---|---|
| Igor Medvedev | 10 | 10 | 10 | 2 | 32 | Gold medal |
| Nikola Pavlović | 2 | 10 | 9 | 0 | 21 | Bronze medal |
| Aleksa Milojević | 10 | 10 | 10 | 10 | 40 | Gold medal |
| Aleksa Konstantinov | 10 | 10 | 9 | 0 | 29 | Bronze medal |
| Ognjen Tošić | 10 | 10 | 10 | 0 | 30 | Silver medal |
| Nikola Sadovek | 10 | 10 | 9 | 0 | 29 | Bronze medal |

Due to three almost equally easy problems, the final results were rather close. All in all, 12 contestants (7 official + 5 guests) with 32-40 points were awarded gold medals, 20 (12+8) with 30-31 points were awarded silver medals, and 37 (21+16) with 17-29 points were awarded bronze medals.

Here is the (unofficial) team ranking:

| Member Countries | | Guest Teams | |
|---|---|---|---|
| 1. Serbia | 181 | Kazakhstan | 181 |
| 2. Romania | 180 | United Kingdom | 152 |
| 3. Turkey | 172 | Italy | 150 |
| 4. Bulgaria | 170 | Saudi Arabia | 145 |
| 5. Greece | 161 | France | 106 |
| 6. Bosnia and Herzegovina | 129 | Azerbaijan | 71 |
| 7. Moldova | 117 | Turkmenistan | 64 |
| 8. Cyprus | 69 | Albania B | 14 |
| 9. Macedonia, FYR | 39 | | |
| 10. Montenegro | 30 | | |
| 11. Albania | 27 | | |

# BALKAN MATHEMATICAL OLYMPIAD

**Tirana, Albania , 07.05.2016.**

**1.** Find all injective functions $f : \mathbb{R} \to \mathbb{R}$ such that for every real number $x$ and positive integer $n$,

$$\left|\sum_{i=1}^{n} i\left(f(x+i+1) - f(f(x+i))\right)\right| < 2016.$$

*(Macedonia, FYR)*

**2.** Let $ABCD$ be a cyclic quadrilateral with $AB < CD$. Its diagonals $AC$ and $BD$ meet at points $F$, and lines $AD$ and $BC$ meet at point $E$. Let $K$ and $L$ respectively be the orthogonal projections of $F$ onto the lines $AD$ and $BC$, and let $M$, $S$ and $T$ respectively be the midpoints of segments $EF$, $CF$ and $DF$. Prove that the second intersection point of circumcircles of triangles $MKT$ and $MLS$ lies on the segment $CD$. *(Greece)*

**3.** Find all monic polynomials $f$ with integer coefficients having the following property: There is a positive integer $N$ such that $2(f(p)!) + 1$ is divisible by $p$ for every prime number $p > N$ for which $f(p)$ is positive.

*Remark:* A polynomial is monic if its leading coefficient is 1. *(Greece)*

**4.** The plane is divided into unit squares by two sets of parallel lines, forming an infinite grid. Each unit square is colored with one of 1201 colors so that no rectangle with perimeter 100 contains two squares of the same color. Show that no rectangle of size $1 \times 1201$ (or $1201 \times 1$) contains two squares of the same color.

*Remark:* Any rectangle is assumed here to have sides contained in the lines of the grid. *(Bulgaria)*

Time allowed: 270 minutes.
Each problem is worth 10 points.

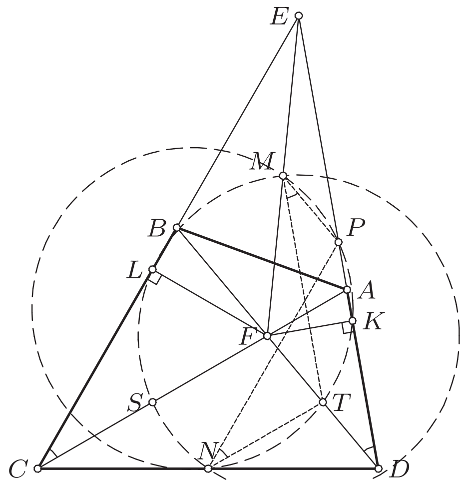
## SOLUTIONS

**1.** Denote $S(x,n) = \sum_{i=1}^{n} i\{f(x+i+1) - f(f(x+i))\}$. Fix $x \in \mathbb{R}$. Subtracting the inequalities $|S(x-n,n)| < 2016$ and $|S(x-n,n-1)| < 2016$ we obtain

$$n \cdot |f(x+1) - f(f(x))| = |S(x-n,n) - S(x-n,n-1)| < 4032 \quad \text{for each } n.$$

This is possible only if $f(f(x)) = f(x+1)$, which by injectivity implies $f(x) = x+1$.

**2.** We claim that the circumcircles of triangles $MKT$ and $MLS$ both pass through the midpoint of segment $CD$.




The circumcircle of $\triangle MKT$ is the nine-point circle of triangle $DEF$, so it also passes through the midpoint $P$ of segment $DE$. Since $PN \parallel EC$, $MT \parallel ED$, $MP \parallel BD$ and $TN \parallel AC$, we have $\sphericalangle PMT = \sphericalangle BDA = \sphericalangle BCF = \sphericalangle PNT$ in oriented angles, so $N$ lies on the circle $MKPT$. Analogously, $N$ lies on the circle $MLS$.

*Remark.* In the case when $CD$ is a diameter of the circumcircle of $ABCD$, the circles $MKT$ and $MLS$ will in fact coincide.

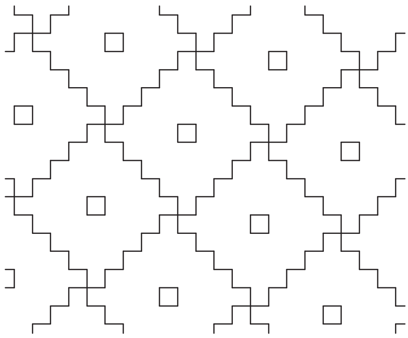
**3.** Clearly, polynomial $f$ is not constant. On the other hand, the condition of the problem implies that $p \nmid f(p)!$, so $f(p) < p$ for all $p > N$. Thus we must have $\deg f = 1$, which means that $f(x) = x - c$ for some $c \in \mathbb{N}$. Wilson's theorem gives us

$$2(p-c)! \equiv -1 \equiv (p-1)! \equiv (-1)^{c-1}(c-1)!(p-c)! \pmod{p}$$

for every prime number $p > N$, so we have $(-1)^{c-1}(c-1)! \equiv 2 \pmod{p}$, i.e, $(-1)^{c-1}(c-1)! = 2$. Hence $c = 3$ and $f(x) = x - 3$.

**4.** We assume that the centers of the square cells are the integer points. We define the *diamond* centered at $(a,b)$ as the set of all cells whose centers $(x,y)$ satisfy $|x-a| + |y-b| \leqslant 24$. Every two cells of a diamond belong to a rectangle with perimeter 100. Since a diamond consists of $24^2 + 25^2 = 1201$ cells, these must include each of the 1201 colors exactly once.

Fix a color - say, blue. Note that every cell $A$ belongs to at least one diamond centered in a blue cell - indeed, the diamond centered at $A$ contains a blue cell,



say $B$, which implies that $A$ belongs to the diamond centered at $B$. On the other hand, if any two diamonds centered at blue cells $B$ and $C$ share a cell $A$, then the diamond centered at $A$ contains both cells $B$ and $C$, which is impossible. Therefore the diamonds with blue centers cover every cell exactly once.

It is easily checked that tiling the plane with diamonds is unique up to symmetry (the image shows an analogous tiling with smaller diamonds). Without loss od generality, the centers of these diamonds are the points $(x, y)$ such that $1201 \mid 24x - 25y$. Since every $x \in \mathbb{Z}$ determines a unique $y$ modulo 1201, the problem statement follows.

The SMO was sponsored by NIS GAZPROM NEFT